LINEAR PROGRAMMING

Formulation

Linear Programming

Linear programming deals with optimization (*max or min*) of linear functions subject to linear constraints.

Three components of a decision making problem

i. Defining of the *decision variables* of the problem

ii. Identification of the *constraints* under which the decision is to be made

iii. Constructing the *objective function* to be optimized – maximizing profit or minimizing cost

General Format of LP Model

Maximize or minimize **Objective Function**

Subject to:

Constraints

And non-negativity of decision variables

- Values of the decision variables that satisfy all the constraints including non-negativity, constitute a feasible solution.
- If the feasible solution maximizes the profit or minimizes the cost it is an optimum feasible solution.

Sample: Linear Programming

Example: Max $z = x + y$

Subject to: $x + 2y \leq 90$

$2x + y \leq 60$

$x \geq 0$

$y \geq 0$

x and y are called **decision variables**

(x + y) is called the **Objective function**

The inequalities are **constraints**

It is called **Linear programming** as the functions are linearly related.

Problem Formulation

- Problem formulation or modeling is the process of translating the verbal statement of a problem into a mathematical statement.
- Formulating models is an art that can only be mastered with practice and experience
- The accuracy and value of the conclusions arrived at depends on how well a model represents the real situation

Formulation of an LPP

- Identify the **Decision Variables** of interest and express them as x_1, x_2, x_3, …
- Ascertain the **Objective** of the problem
- Ascertain the **Cost** (in case of minimization) or the **Profit** (in case of maximization) per unit of each decision variable
- On the basis of the above data, write down the **Objective Function**, Z, as a linear function of the decision variables

LPP Formulation (contd.)

- Ascertain the **Constraints** representing the maximum availability (of resources) or the minimum commitment (demands, targets etc.) or equality

- Write the constraints in terms of decision variables as “less than or equal to” (<=) type inequality or “greater than or equal to” (>=) type inequality or “equal to” (=) type equality; all constraints shall be linear

- Note that maximum availability leads to a <= type and minimum commitment gives a >= type inequality

Formulation (contd.)

- Add **non-negativity** restriction as under:

  $x_j >= 0; j = 1, 2, \dots n$

Non-negativity constraints are a general feature of all LPPs

The Mathematical Formulation

The Mathematical Formulation looks like:

Maximize (or Minimize) $Z = c_1x_1 + c_2x_2 + \ldots$

Subject to constraints:

$a_{11}x_1 + a_{12}x_2 + \ldots <= b_1$ (Maximum availability)

$a_{21}x_1 + a_{22}x_2 + \ldots >= b_2$ (Minimum commitment)

$a_{31}x_1 + a_{32}x_2 + \ldots = b_3$ (Equality)

…

$x_1, x_2, \ldots >= 0$ (Non-negativity restriction)

A wide range of LP Problems

- “Product Mix” Problems (e.g. 1, 2, 3)
- “Make or Buy Decision” Problems (e.g. 4)
- “Choice of Alternatives” Problems
- “Sales Budget” Problems
- “Production Budget” Problems
- “Purchase Budget” Problems
- “Portfolio Mix” Problems (e.g. 5)
- “Advertising” Problem

Range of LP Problems (contd.)

- “Capital Mix” Problems
- “Diet” Problems
- “Nutrition” Problems (e.g. 6)
- “Blending” Problems
- “Trim” Problems
- “Transportation” Problems
- “Assignment Problems”
- “Job Scheduling” Problems

Example-1

A garment manufacturer has a production line making two styles of shirts. Style I needs 200 g of cotton thread, 300 g of Dacron thread and 300 g of linen thread. Corresponding requirements of style II are 200g, 200g and 100g. The net contributions are Rs.19.50 for style I and Rs.15.90 for style II. The available inventory of cotton thread, Dacron thread and linen thread are, respectively, 24 kg, 26 kg and 22 kg.

The manufacturer wants to determine the number of each style to be produced with the given inventory. Formulate the LPP model.

Step 1: Objective Function

- Decision variables: These are the numbers of each style to be produced:
  - Number of style I shirts, say x_1
  - Number of style II shirts, say x_2

- Objective of the decision maker: Maximize total contribution , given that the contribution per unit is Rs. 19.50 for x_1 and Rs. 15.90 for x_2
- Maximize profit
- Hence the objective function is:

$$\text{Max } Z = 19.50\, x_1 + 15.90\, x_2$$

Step 2: Constraints

- Availability of cotton thread: Style I needs 200 g and style II needs 200 g. 24,000 g is available. The corresponding constraint is

  $$200\ x_1 + 200\ x_2 <= 24{,}000$$

- Availability of dracon thread: Style I needs 300 g and style II needs 200 g. 26,000 g is available. The corresponding constraint is

  $$300\ x_1 + 200\ x_2 <= 26{,}000$$

- Availability of linen thread: Style I needs 300 g and style II needs 100 g. 22,000 g is available. The corresponding constraint is

  $$300\ x_1 + 100\ x_2 <= 22{,}000$$

Step 3: Non-negativity

Since the number of shirts cannot be negative, add the non-negativity restriction $x_1, x_2 >= 0$ and complete the LP formulation.

Note: For the time being we ignore another restriction on x_1 and x_2 that they be integers. If that restriction is also added we write:

$x_1, x_2 >=0$ and integers. Then this becomes an Integer LP or ILP, which we will see later.

Solution of Example 1

Let x_1 = number of style I shirts

x_2 = number of style II shirts

Max $Z = 19.50\ x_1 + 15.90\ x_2$ (contribution)

Sub to: $200\ x_1 + 200\ x_2 <= 24{,}000$ (cotton)

$300\ x_1 + 200\ x_2 <= 26{,}000$ (dacron)

$300\ x_1 + 100\ x_2 <= 22{,}000$ (linen)

$x_1, x_2 >= 0$ (non-negativity)

Example 2

An animal feed company must produce 200 kg of a mixture consisting of ingredients A and B daily. A costs Rs. 3 per kg and B costs Rs. 8 per kg. Not more than 80 kg of A can be used and at least 60 kg of B must be used.

The company wants to know how much of each ingredient should be used to minimize cost. Formulate the LPP.

Step 1: Objective Function

- Decision variables: Qty of each ingredient used:
  - Quantity of ingredient A, say A kg
  - Quantity of ingredient B, say B kg

- Objective of the decision maker: Minimize cost, given that cost per kg is Rs. 3 for A; Rs. 8 for B.

- Hence the objective function is:

  Min $Z = 3A + 8B$

Step 2: Constraints

- Committed output is 200 kg of the mixture. The corresponding constraint is

  A + B = 200

- Not more than 80 kg of A can be used. The corresponding constraint is

  A <= 80

- At least 60 kg of B must be used. The corresponding constraint is

  B >= 60

Step 3: Non-negativity

Since the decision variables cannot be negative, add the non-negativity restriction A, B >= 0 and complete the LP formulation.

Note that we do not have any integer restrictions on the decision variables in this case.

Solution to Example 2

$$\text{Min } Z = 3A + 8B$$

Sub to:

$$A + B = 200$$

$$A \le 80$$

$$B \ge 60$$

$$A, B \ge 0$$

Example 3

A farmer has a 125 acre farm. He produces radish, mutter and potato. Whatever he raises is fully sold. He gets Rs.5 per kg for radish, Rs.4 per kg for mutter and Rs.5 per kg for potato. The average yield per acre is 1500 kg for radish, 1800 kg for mutter and 1200 kg for potato. Cost of manure per acre is Rs.187.50, Rs.225 and Rs.187.50 for radish, mutter and potato respectively. Labour required per acre is 6 man-days each for radish and potato and 5 man-days for mutter. A total of 500 man-days of labour is available at the rate of Rs.40 per man-day.

Formulate this as an LPP model to maximize the profit.

Step 1: Objective Function

- Decision variables: acreage for each produce:
  - Acres for radish, say r
  - Acres for mutter, say m
  - Acres for potato, say p

- Objective of the decision maker: Maximize profit, given that, profit per acre is:
  - (5*1500 -187.5 – 6*40) for radish
  - (4*1800 – 225 - 5*40) for mutter
  - (5*1200 – 187.5 – 6*40) for potato

E.g.: Profit per acre for Radish

- Earning per acre
  Each acre yields 1500 kg. Each kg sells for Rs.5
  Hence earning per acre = Rs.1500*5

- Cost per acre
  Manure = Rs.187.50
  Labour = 6 man-days @ Rs.40 per man-day.
  Hence cost per acre is Rs.(187.50 + 6*40)

- Profit = Earning – cost = Rs.(1500*5 – 187.5 – 6*40)

Step 2: Constraints

- Availability of land = 125 acres. The corresponding constraint is

  $r + m + p <= 125$

- Availability of labour = 500 man-days. The corresponding constraint is

  $6r + 5m + 6p <= 500$

Step 3: Non-negativity

Since the decision variables cannot be negative, add the non-negativity restriction r, m, p >= 0 and complete the LP formulation.

Solution to Example 3

Let r, m, p be the no of acres used for radish, mutter and potato respectively.

Max Z = (5*1500 -187.5 – 6*40) r + (4*1800 – 225 - 5*40) m + (5*1200 – 187.5 – 6*40)p

Sub to: r + m + p <= 125 (Land constraint)

6r + 5m + 6p <= 500 (Man-days constraint)

r, m, p >= 0 (non-negativity)

Example 4

Jindal manufactures a type of sofa set containing seven components: one sofa, two centre tables and four chairs.

These can either be manufactured in-house or sub-contracted as per the data given in the table:

Per component	**Sofa**	**Table**	**Chair**
Direct Material	Rs. 1,000	Rs. 500	Rs. 550
Direct Labour hours	100	50	10
Sub-contract price	Rs. 2,500	Rs. 1,000	Rs. 750

Sales of sofa sets are 8,000 per period, each selling for Rs.7,500. A capacity constraint of 500,000 direct labour hours obliges the company to sub-contract some components.

The variable overheads vary with direct labour hours at Rs. 2 per hour. Fixed costs are Rs. 1,750,000 per period and labour costs Rs. 5.50 per hour.

Formulate LPP to minimize costs.

- Decision variables:

  The nos of sofas, tables, chairs made / bought:

  $s_m, s_b, t_m, t_b, c_m, c_b$.
- The table on the next slide calculates the cost per unit of each decision variable (objective function coefficients)
- Hence the objective function is:

  Min $1750\ s_m + 2500\ s_b + 875\ t_m + 1000\ t_b + 625\ c_m + 750\ c_b + FC$

- Constraints:
  - Demand for sofas: $s_m + s_b = 8000$
  - Demand for tables: $t_m + t_b = 16{,}000$
  - Demand for chairs: $c_m + c_b = 32{,}000$
  - Direct labour hours: $100\ s_m + 50\ t_m + 10\ c_m <= 500{,}000$

- Non-negativity: $s_m, s_b, t_m, t_b, c_m, c_b >= 0$

Solution to example 4

	Sofa	Table	Chair
DM cost	1000	500	550
DL cost	550	275	55
Var O/H	200	100	20
Cost of "make"	1750	875	625
Cost of Buy	2500	1000	750

Min $Z = 1750\ s_m + 2500\ s_b + 875\ t_m + 1000\ t_b + 625\ c_m + 750\ c_b + FC$

Subject to: $s_m + s_b = 8000$ (demand for sofas)

$t_m + t_b = 16{,}000$ (demand for table)

$c_m + c_b = 32{,}000$ (demand for chairs)

$100\ s_m + 50\ t_m + 10\ c_m <= 500{,}000$ (direct labour available)

$s_m, s_b, t_m, t_b, c_m, c_b >= 0$ (non-negativity)

Example 5

A mutual fund has Rs. 2 million available for investment in Government bonds, blue chip stocks, speculative stocks and short-term bank deposits. The annual expected return and the risk factor are as shown:

Investment	Return %	Risk factor (0- 100)
Bonds	14	12
Blue Chip	19	24
Speculative	23	48
Short-term	12	6

The fund is required to keep at least Rs. 200,000 in short-term deposits and not to exceed an average risk factor of 42. Speculative stocks must not exceed 20% of the money invested. Formulate the LPP maximizing expected annual return.

- Decision variables:

  Amounts invested in Government bonds, blue chip stocks, speculative stocks and short-term bank deposits :

  x_1, x_2, x_3 and x_4.
- Objective: Maximize returns given that the return per Re. for the investments are 0.14, 0.19, 0.23, 0.12
- Hence the objective function is:

  Max $0.14\ x_1 + 0.19\ x_2 + 0.23\ x_3 + 0.12\ x_4$

- Constraints: (See next slide for calculations)
  - Amount available: $x_1 + x_2 + x_3 + x_4 .<= 2{,}000{,}000$
  - At least 200,000 in STD: $x_4 >= 200{,}000$
  - Average risk factor: $30x_1+18x_2- 6x_3+36x_4 \geq 0$
  - Limit on speculative stock: $0.2\ x_1 + 0.2\ x_2 – 0.8\ x_3 + 0.2\ x_4 \geq 0$

- Non-negativity: $x_1, x_2, x_3, x_4 >= 0$

Solution to Example 5

Let x_1, x_2, x_3, x_4 be the amounts invested.
Average risk factor =
$(12\ x_1+24\ x_2 + 48\ x_3 + 6\ x_4)/(x_1+ x_2+ x_3+ x_4) \leq 42$.
This gives $30x_1+18x_2-6x_3+36x_4\geq 0$.
Also $x_3 \leq 0.2\ (x_1+ x_2+ x_3+ x_4)$ ---- Maximum limit on speculative stock
This gives: $0.2\ x_1 + 0.2\ x_2 – 0.8\ x_3 + 0.2\ x_4 \geq 0$

Hence the LPP formulation is as follows:
Max $Z = 0.14\ x_1+ 0.19\ x_2+ 0.23\ x_3+ 0.12\ x_4$
s.t: $30x_1+18x_2- 6x_3+36x_4\geq 0$ (Avg Risk factor)
$0.2\ x_1 + 0.2\ x_2 – 0.8\ x_3 + 0.2\ x_4 \geq 0$ (limit on speculative stock)
$x_1 + x_2+ x_3 + x_4 \leq 2{,}000{,}000$
$x_4 \geq 200{,}000$
$x_1, x_2, x_3, x_4 \geq 0$ (Non-negativity)

Example 6

The vitamins V and W are found in two different foods, F_1 and F_2. The respective prices per unit of each food are Rs. 3 and Rs. 2.5. One unit of F_1 contains 2 units of vitamin V and 3 units of vitamin W. One unit of F_2 contains 4 units of vitamin V and 2 units of vitamin W. The daily requirements of V and W are at least 60 units and 75 units respectively.

Formulate an LPP to meet the daily requirement of the vitamins at minimum cost

- Decision variables:

  Quantity of foods F_1 and F_2 required be x_1 and x_2.
- Objective: Minimize cost given that the costs per unit of F_1 and F_2 are Rs. 3 and Rs. 2.5 respectively
- Hence the objective function is:

  Min $Z = 3\ x_1 + 2.5\ x_2$

- Constraints:
  - Requirement of V: $2x_1 + 4\ x_2 \geq 60$
  - Requirement of W: $3x_1 + 2\ x_2 \geq 75$

- Non-negativity: $x_1, x_2 >= 0$

Solution to Example 6

Let x_1 and x_2 be the quantities of F_1 and F_2.

Minimise: $3x_1 + 2.5\ x_2$

s.t: $2x_1 + 4\ x_2 \geq 60$ (Min requirement of V)

$3x_1 + 2x_2 \geq 75$ (Min requirement of W)

$x_1, x_2 \geq 0$